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Refined topological vertex and instanton counting

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ABSTRACT: It has been proposed recently that topological A-model string amplitudes for toric Calabi-Yau 3-folds in non self-dual graviphoton background can be caluculated by a diagrammatic method that is called the "refined topological vertex". We compute the extended A-model amplitudes for SU(N)-geometries using the proposed vertex. If the refined topological vertex is valid, these computations should give rise to the Nekrasov's partition functions of $\mathcal{N} = 2$ SU(N) gauge theories via the geometric engineering. In this article, we verify the proposal by confirming the equivalence between the refined A-model amplitude and the K-theoretic version of the Nekrasov's partition function by explicit computation.

KEYWORDS: Topological Strings, Supersymmetric gauge theory.

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Contents

1. Introduction

The study of topological A-model strings on non-compact toric Calabi-Yau manifolds has been the important subject in the research of topological strings. Topological strings has provided insights into mathematics and nonperturbative dynamics of gauge and string theory.

On the one hand, it is in general very hard to caluculate the topological string partition functions exactly. However in some cases, various dualities enable us to simplify the caluculation of topological strings greatly and provide new perspectives [1]. For example, the A-model partition function on the resolved conifold is given by the partition function of Chern-Simons theory on S^3 . This is the geometric transition between the resolved conifold and the deformed conifold [2 − 4]. By generalizing this argument, an elegant technique for computing the A-model partition function on toric Calabi-Yau manifolds was formulated in [5]. The formalism is called the topological vertex.

The mechanism of the geometric engineering is one way to study supersymmetric gauge theories using string theory and topological string [6]. This approach tells us that we can caluculate the F-terms of various $\mathcal{N}=2$ SU(N) gauge theories by using topological A-model strings on certain toric Calabi-Yau manifolds. The partition function of the A-model on the toric Calabi-Yau agrees with the Nekrasov's partition function of $\mathcal{N}=2$ SU(N) gauge theory [7 − 13]. Thus topological strings are useful tool to obtain insights into the nonperturbative dynamics of supersymmetric gauge theories.

The Nekrasov's partition function in a constant self-dual graviphoton background contains] one parameter which is corresponding to the value of the background field. The parameter is nothing but the topological string coupling constant in A-model side. On the other hand, we can perform the instanton caluclation in the more general background of non self-dual graviphoton configuration, and we get the K-theoretic version[1] of the Nekrasov's partition function [7, 14, 15]. Then the Nekrasov's partition function has one more parameter in addition to the self-dual graviphoton background. Hence it is natural to expect that there exists a 2-parameter extension of the topological vertex which will recover the K-theoretic answer. Few attempts were made for defining the 2-parameter extension of topological strings and formulating the algorithmical techniques to caluculate the extended partition function [11, 16, 17]. Recently a refined topological vertex was proposed in [18]. In this article, we compute the refined topological A-model string partition function for the SU(N) geomerties and check the equivalence of the refined partition function and the K-theoretic version of the Nekrasov's partition function.

This paper is organized as follows. In section 2, we review the geometric engineering, the topological vertex and their 2-parameter extension. The refined A-model partition function for SU(N) geomerties are calculated and the modification of the framing factor is proposed in section 3. Conclusions are found in section 4. In appendix A, we give brief introduction to Young diagrams, Schur functions, and the useful formulae for Schur functions. In appendix B, a proof of a formula can be found.

2. Topological strings and instanton counting

In this section, we will briefly review the idea of the geometric engineering, topological A-model strings, and the instanton counting.

2.1 Geometric engineering and A-model

Type IIA string theory compactified on a Calabi-Yau 3-fold yields an effective theory in transverse 4-dimensions. Especially, enhanced gauge symmetries arise from singular Calabi-Yau compactification. Thus in the field theory limit, appropriate Calabi-Yau compactifications provide effective gauge theories in 4-dimensions. This is the basic idea of the geometric engineering [6].

Let us consider Type IIA compactified on a Calabi-Yau 3-fold M . The Kähler parameters of M are denoted by t_i. Then, the F-term of the effective theory is given by [19, 20]

$$\sum_{g=0}^{\infty}\int d^4\,x\,d^4\,\theta\,W^{2g}\,F_g(t_i)=\int d^4\,x\left[\tau_{ij}\,F^i_{\mu\nu}\,F^{j\mu\nu}+\sum_{g=1}^{\infty}F_g(t_i)\,{R_+}^2\,{F_+}^{2g-2}\right]\tag{2.1}$$

[1]These partition functions are called K-theoretic because they are obtained from K-theoretic localization.

Here, W is $W_{\mu\nu} = F^{+}_{\mu\nu} - R_{\mu\nu\rho\sigma}\,\theta\,\sigma^{\rho\sigma}\,\theta + \cdots$, F_+ is the self-dual part of the graviphoton field strength, R_+ is the self-dual part of the Riemann tensor, and $F^i_{\mu\nu}$ is the $\mathrm{U}(1)$ gauge field strength of the effective theory. Notice that the 4-dimensional $\mathrm{U}(1)$ gauge couplings are given by

$$\tau_{ij} = \frac{\partial^2}{\partial\, t_i\, \partial\, t_j}\, F_0(t_i) \tag{2.2}$$

Hence the genus zero amplitudes of Type IIA strings $F_0(t_i)$ give rise to the effective gauge couplings. This is the Seiberg-Witten theory [21] in Type IIA string theory set-up. The higher genus amplitudes $F_g(t_i)$ correspond to the graviphoton corrections to the gauge theory. They play an important role in the Nekrasov's partition function that gives a closed expression for the Seiberg-Witten prepotential [7, 14, 15].

Furthermore, the amplitudes of Type IIA strings $F_g(t_i)$ are identical with the topological A-model string amplitudes $\mathcal{F}_g(t_i)$ of M which "count" the holomorphic maps from genus g Riemann surfaces to a Calabi-Yau M [19, 20]. The information of the partition function was encoded in the Gromov-Witten invariants. The generating function of these amplitudes is called the topological A-model string partition function

$$Z = \exp\left(\mathcal{F}(g_s, t_i)\right) = \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2}\mathcal{F}_g(t_i)\right) \tag{2.3}$$

Here g_s is the topological string couplins constant.

2.2 Gopakumar-Vafa invariants

The target space perspective tells us that we can reformulate A-model as BPS state counting problem. Let us consider M-theory lift of Type IIA on a Calabi-Yau, i.e. M-theory compactified on a Calabi-Yau times a circle. This set-up gives rise to an effective field theory in the transverse 5-dimension $\mathbb{R}^{1,3} \times \mathrm{S}^1$. The particles in the effective theory arise from M2 branes wrapping holomorphic curves of M. The mass and the charge (j_L, j_R) of the little group in 5-dimensions $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ characterise these BPS particles. The masses are given by $m_{(\Sigma,n)} = T_\Sigma + \frac{2\pi i n}{g_s}$. Here T_Σ is the Kähler parameter of the curve class Σ which M2 brane wraps, and n is the momentum along S^1. Therefore the mass (and cherge via BPS condition) is given by the curve class Σ and the momentum n. Integrating out these particles, we get the F-term of the effective theory [3]

$$\begin{aligned}
\mathcal{F} &= \sum_{\Sigma\in H_2(M,\mathbb{Z})}\sum_{n\in Z}\sum_{j_L,j_R} N_\Sigma^{(j_L,j_R)} \log\det{}_{(j_L,j_R)}\left(\Delta + {m_{(\Sigma,n)}}^2 + 2m_{(\Sigma,n)}\,\sigma_L\, F_+\right)\\
&= \sum_{\Sigma\in H_2(M,\mathbb{Z})}\sum_{k=1}^{\infty}\sum_{j_L} N_\Sigma^{j_L}\, e^{-k\,T_\Sigma}\,\frac{Tr_{j_L}\,(-1)^{\sigma_L}\, e^{-2k\,g_s\,\sigma_L}}{k\,(2\sinh(k\,g_s\,/2))^2}\\
&= \sum_{\Sigma\in H_2(M,\mathbb{Z})}\sum_{k=1}^{\infty}\sum_{j_L} N_\Sigma^{j_L}\,(-1)^{-2\,j_L}\, e^{-k\,T_\Sigma}\,\frac{\sum_{l=-\,j_L}^{j_L} q^{-2kl}}{k\left(q^{k/2}-q^{-k/2}\right)^2}
\end{aligned}$$

Notice that the graviphoton expectation value gives topological string coupling $F_+ = g_s$ and we introduce $q = e^{-g_s}$. Changing representation basis of $\mathrm{SU}(2)_L$ so as to satisfy $\sum_{j_L} N_\Sigma^{j_L}(\sum_{l=-j_L}^{j_L} q^l) = \sum_{g=0}^{\infty} n_\Sigma^g (-1)^g (q^{1/2}-q^{-1/2})^{2g}$, we get the following expression of the A-model partition function

$$\mathcal{F} = \log Z = \sum_{\Sigma \in H_2(M,\mathbb{Z})} \sum_{k=1}^{\infty} \sum_{g=0}^{\infty} \frac{n_\Sigma^g}{k} (q^{k/2}-q^{-k/2})^{2g-2} e^{-T_\Sigma} \tag{2.4}$$

Integer valued invariants n_Σ^g which are defined as above are called "Gopakumar-Vafa invariants".

$N_\Sigma^{(j_L,j_R)}$ is the number of the wrapped M2-branes, and they are not invariant under the complex structure deformations of Calabi-Yau. Roughly speaking, this is the reason why the information encoded in the partition function is not the full degeneracies $N_\Sigma^{(j_L,j_R)}$ but $N_\Sigma^{j_L}$ which are summed over $\mathrm{SU}(2)_R$ charges as

$$N_\Sigma^{j_L} = \sum_{j_R} (-1)^{-2j_R}(2j_R+1)\, N_\Sigma^{(j_L,j_R)} \tag{2.5}$$

However $N_\Sigma^{(j_L,j_R)}$ themselves are invariants for non-compact Calabi-Yau since these Calabi-Yau 3-folds have no complex structure deformations [11]. Among them, local toric Calabi-Yau 3-folds are important ones. Hence we define an extended partition function that counts invariants $N_\Sigma^{(j_L,j_R)}$ as follows

$$\begin{aligned}\mathcal{F} &= \sum_{\Sigma \in H_2(M,\mathbb{Z})} \sum_{n \in Z} \sum_{j_L,j_R} N_\Sigma^{(j_L,j_R)} \log \det{}_{(j_L,j_R)} \left(\Delta + {m_{(\Sigma,n)}}^2 + 2m_{(\Sigma,n)}\, \sigma_L\, (F_+ + F_-)\right) \\ &= \sum_{\Sigma \in H_2(M,Z)} \sum_{k=1}^{\infty} \sum_{j_L,j_R} N_\Sigma^{(j_L,j_R)} (-1)^{-2(j_L+j_R)} e^{-k\,T_\Sigma} \frac{\left(\sum_{l=-j_L}^{j_L} (tq)^{-2kl}\right)\left(\sum_{m=-j_R}^{j_R} \left(\frac{t}{q}\right)^{-2km}\right)}{k\left(t^{k/2}-t^{-k/2}\right)\left(q^{k/2}-q^{-k/2}\right)}\end{aligned}$$

Here $q = e^{F_+}$ and $t = e^{F_-}$.

The question now arises; how to compute these partition functions for non-compact Calabi-Yau. In the case of toric Calabi-Yau 3-folds, the answer can be found in a diagrammatic methods named the topological vertex. Before we turn to the discussion of topological vertex, it will be useful to take a look at the instanton counting of $\mathcal{N}= 2$ gauge theory. Hence in the next section, we discuss the Nekrasov's partition function of $\mathcal{N}= 2$ gauge theory. We will come back to the discussion of the topological vertex later.

2.3 Instanton counting of $\mathcal{N} = 2$ gauge theories

Instanton calculation of $\mathcal{N}= 2$ gauge theories in 4- and 5-dimensions has been developed by Nekrasov [7]. He found that the instanton coefficients of the Seiberg-Witten prepotential are summed up to a closed form, and he provided the combinatorical expression of this generating function. We call it the Nekrasov's partition function. His conjectual observation was mathematically verified by Nekrasov-Okounkov [23], Nakajima-Yoshioka [24], and Braverman [25].

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Take an $\mathcal{N}= 2$ SU(N) supersymmetric pure Yang-Mills theory for example. Muiti-instanton calculation involves an integral over the ADHM moduli space. It is in general very hard to carry out the caluculation. However we can formulate the muiti-instanton calculation of $\mathcal{N}= 2$ SU(N) supersymmetric gauge theory as integrals of equivariant closed forms. Let us consider the following partition function of $\mathcal{N}= 2$ SU(N) supersymmetric pure Yang-Mills theory

$$Z^{inst.}(\vec{a},\Lambda) = \sum_{k=1}^{\infty} \Lambda^{2Nk}\, Z^k(\vec{a})$$

Here $\vec{a}$ ia the Coulomb moduli, Λ is the dynamical scale, and $Z^k(\vec{a})$ is a k-instanton contribution. By deforming the theory by torus action on the moduli space, we can give the partition function as an integral of equivariant differential

$$Z^k = \int_{\mathcal{M}(N,k)} \mathcal{D}\mu\, e^{-Q\Psi} \tag{2.6}$$

where $\mathcal{M}(N,k)$ is the ADHM moduli space of k-instantons and Q is the BRST operator. It is known that the BRST operator is an equivariant differential for torus action $T = \mathrm{U}(k) \times \mathrm{U}(1)^{N-1} \times \mathrm{U}(1)^2$ on the moduli space. Here $\mathrm{U}(1)^2$ is the rotation groups of complex plane $\mathbb{R}^4 = \mathbb{C}^2$ and their weights provide deformation parameters ϵ_i. Then we can apply the localization formula

$$Z^k = \sum_{p_0} \frac{1}{\sqrt{\det \mathcal{L}_{p_0}}} \tag{2.7}$$

Here p_0 are isolated fixed points of the torus action and $\mathcal{L}_{p_0}$ is the Lie derivative acting on the tangent moduli space $T\mathcal{M}(N,k)$. It is known that the fixed points of T-action are uniquely specified by N Young diagrams $(\mu_1,\ldots,\mu_N)$. Then we have to know the weights $\det \mathcal{L}_{p_0}$ of T-action on the tangent moduli space $T\mathcal{M}(N,k)$ for the purpose of multi-instanton caluculus. The weights were caluculated in [26, 7, 14, 15] and the explicit expression is given by

$$\begin{aligned} Z^{inst.}(\epsilon_1,\epsilon_2,\vec{a},\Lambda) = \sum_{\vec{\mu}} \Lambda^{2N|\vec{\mu}|} \prod_{a,b=1}^{N} \prod_{s\in\mu_a} \frac{1}{a_b - a_a - \epsilon_1\, l_{\mu_b}(s) + \epsilon_2(a_{\mu_a}(s)+1)} \\ \times \prod_{t\in\mu_b} \frac{1}{a_b - a_a + \epsilon_1(l_{\mu_a}(s)+1) - \epsilon_2\, a_{\mu_b}(s)} \end{aligned} \tag{2.8}$$

Nekrasov claimed that the partition function (2.8) leads to the Seiberg-Witten prepotential after eliminating the deformation parameter ϵ as follows

$$\epsilon_1\epsilon_2 \log Z^{inst.}(\epsilon_1,\epsilon_2,\vec{a},\Lambda) = \mathcal{F}^{inst.}_{\mathrm{SW}}(\vec{a},\Lambda) + O(\epsilon_1,\epsilon_2) \tag{2.9}$$

This conjecture was proved by using the thermodynamical limit of the random partition [23], the blow-up equation [24], and [25].

We can lift it to the 5-dimensional gauge theory result

$$Z_{5D}^{inst.}(t,q,\vec{a},\Lambda,\beta)=\sum_{\vec{\mu}}(\beta\Lambda)^{2N|\vec{\mu}|}\prod_{a,b=1}^{N}\prod_{s\in\mu_a}\frac{1}{1-Q_{ba}\,t^{l_{\mu_b}(s)}\,q^{a_{\mu_a}(s)+1}}\prod_{t\in\mu_b}\frac{1}{1-Q_{ba}\,t^{-\,l_{\mu_a}(s)-1}\,q^{-\,a_{\mu_b}(s)}} \tag{2.10}$$

Here β is the radius of the compact fifth dimension S^1. Let us choose the deformation parameter ϵ as $\hbar=\epsilon_1=-\epsilon_2$. In [10, 9, 11] the partition function was reproduced from the string calculation via the geometric engineering

$$Z_{5D}^{Nek.\,\mathrm{SU}(N)}(\hbar,-\hbar,\vec{a},\Lambda,\beta)=Z^{A-model,\mathrm{SU}(N)}(\hbar=g_s,t_i) \tag{2.11}$$

and they verified the interpretation in [7] that $\hbar$ expansion is nothing but the genus expansion of the string partition function. Notice that the Coulomb moduli $\vec{a}$ and the dynamical scale Λ are engineered from the Kähler parameters of the Calabi-Yau. We review the results (2.11) for SU(2) theory later.

Thus it is natural to expect that there is a refinement of string theory to engineer Nekrasov' partition function for the general case $\epsilon_1\neq-\epsilon_2$. In this paper we calculate the K-theoretic partition function (2.10) via the refined topological vertex and show that the refined A-model of [18] reproduces the correct results.

2.4 Topological vertex and its refinement

It is known that we can compute the topological A-model string amplitudes for toric Calabi-Yau 3-folds by using the topological vertex [5]. The topological vertex is the Feymnan-rules like technique which arises from the geometric transition between A-model and Chern-Simons gauge theory. The Feymnan diagrams, the vertices of diagrams, the momentun, and the propagators are corresponding to the toric web-diagrams, the tri-valent vertices $C_{\mu_1\,\mu_2\,\mu_3}$, Young diagrams μ, and the weights $(-1)^{(n+1)|\mu|}\,e^{-T|\mu|}\,q^{-\frac{n\,\kappa_\mu}{2}}$, respectively. Here, T is the Kähler parameter for the 2-cycle corresponding to the line of the web-diagram, μ is the Young diagram which propagates along the line. The framing number n is determined by the toric diagram. The vertex is expressed using the Schur functions

$$C_{\lambda\mu\nu}(q)=q^{\kappa_\mu/2}\,s_{\nu^t}(q^{-\rho})\sum_{\eta}s_{\lambda^t/\eta}(q^{-\nu-\rho})\,s_{\mu/\eta}(q^{-\,\nu^t-\rho}) \tag{2.12}$$

See appendix A for the definition and properties of the Schur functions. The vertices in figure 1 are glued as

$$\sum_{\nu}C_{\lambda\mu\nu}(q)\,(-1)^{(n+1)|\nu|}\,q^{-n\kappa_\nu/2}\,e^{-T|\nu|}\,C_{\lambda'\mu'\,\nu^t}(q) \tag{2.13}$$

where the framing number n is given by $n=v'\wedge v={v'}_1\,v_2-v_1\,{v'}_2$.

The local Hirzebruch surface $\mathbb{F}_0=C(\mathbb{P}^1\times\mathbb{P}^1)$ is a good example to illustlate the topological vertex calculation. This toric Calabi-Yau 3-fold is the typical SU(2) geometry that

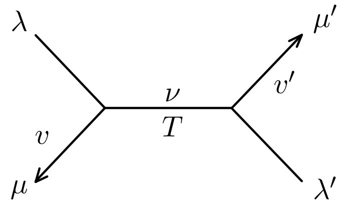

Figure 1: The toric diagram obtained by gluing the vertices $C_{\lambda\mu\nu}$ and $C_{\lambda'\mu'\nu^t}$

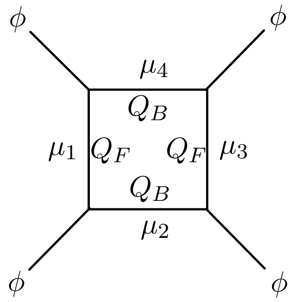

Figure 2: The local Hirzebruch surface which is a line bundle over $\mathbb{P}^1 \times \mathbb{P}^1$

engineers SU(2) pure super Yang-Mills theory. The toric diagram is given by figure 2, and we can easily check that the framing numbers associated to the four internal lines are all 1. Appling the topological vertex to figure 2, we get the following partition function

$$\begin{aligned} Z^{F_0}(Q_F, Q_B) &= \sum_{\mu_1,\mu_2,\mu_3,\mu_4} {Q_F}^{|\mu_1|+|\mu_3|}\, {Q_B}^{|\mu_2|+|\mu_4|}\, q^{-\kappa_{\mu_1}/2+\kappa_{\mu_2}/2-\kappa_{\mu_3}/2-\kappa_{\mu_4}/2} \\ &\qquad \times C_{\phi\,\mu_1\,{\mu_4}^t}\, C_{\phi\,{\mu_2}^t\,{\mu_1}^t}\, C_{\mu_2\,\phi\,\mu_3}\, C_{\phi\,\mu_4\,{\mu_3}^t} \\ &= \sum_{\mu_2,\mu_4} {Q_B}^{|\mu_2|+|\mu_4|}\, q^{+\kappa_{\mu_2}/2-\kappa_{\mu_4}/2}\, K_{\mu_4\,\mu_2}(Q_F)\, K_{{\mu_2}^t\,{\mu_4}^t}(Q_F) \end{aligned}$$

$K_{\mu\nu}$ is defined as follows

$$\begin{aligned} K_{\mu\nu} &= \sum_\lambda {Q_F}^{|\lambda|}\, q^{-\kappa_\lambda/2}\, C_{\phi\lambda\,\mu^t}\, C_{\nu^t\,\lambda^t\,\phi} \\ &= s_{\mu^t}(q^{-\rho})\, s_\nu(q^{-\rho}) \sum_\lambda {Q_F}^{|\lambda|}\, s_\lambda\left(q^{-\mu-\rho}\right) s_\lambda\left(q^{-\nu^t-\rho}\right) \\ &= q^{||\mu||^2/2+||\nu^t||^2/2}\, \tilde{Z}_{\mu^t}(q)\, \tilde{Z}_\nu(q) \prod_{i,j=1}^{\infty} \frac{1}{1-Q_F\, q^{-\mu_i-{\nu^t}_j+i+j-1}} \end{aligned}$$

where we use the relation $s_\mu(q^{-\rho}) = q^{||\mu^t||^2/2} \prod_{s\in\mu} \left(1-q^{h_\mu(s)}\right)^{-1} = q^{||\mu^t||^2/2}\, \tilde{Z}_\mu(q)$ and formula (A.17). Let us separate out the perturbative contributions as

$$Z^{\mathbb{F}_0}(Q_B, Q_F) = Z^{\mathbb{F}_0}_{pert.}(Q_F)\, Z^{\mathbb{F}_0}_{inst.}(Q_B, Q_F) \tag{2.14}$$

$$Z^{\mathbb{F}_0}_{pert.}(Q_F) \equiv K_{\phi\phi}(Q_F)^2 = \left[\prod_{i,j=1}^{\infty} \frac{1}{1-Q_F\, q^{i+j-1}}\right]^2 \tag{2.15}$$

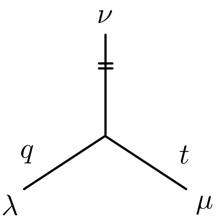

Figure 3: The refined topological vertex $C_{\lambda\mu\nu}(t,q)$

Then, we get the A-model partition function corresponding to the nonperturbative part of the Nekrasov's partition function

$$Z^{\mathbb{F}_0}_{inst.} = \sum_{\mu,\nu} Q_B{}^{|\mu|+|\nu|}\, q^{\|\mu\|^2+||\nu^t||^2}\, \tilde{Z}_\mu(q)\, \tilde{Z}_{\mu^t}(q)\, \tilde{Z}_\nu(q)\, \tilde{Z}_{\nu^t}(q) \left[\prod_{i,j=1}^{\infty} \frac{1-Q_F\, q^{+i+j-1}}{1-Q_F\, q^{-\mu_i-\nu^t{}_j+i+j-1}}\right]^2 \tag{2.16}$$

In fact, appling the formula (3.20) for the special case we can show that the above result is identical with the Nekrasov's partition function of the SU(2) Yang-Mills theory (2.10) for $t=q$. The identifications of parameters are given by

$$Q_B = (\beta\Lambda)^4, \quad Q_F = e^{2\beta a} \tag{2.17}$$

Recently, the topological vertex formalism for the refined partition functions has been proposed in [18] via melting crystal picture of the topological vertex. We call it the refined topological vertex. It was claimed that the refined topological vertex is constructed so as to engineer the K-theoretic version of the Nekrasov's partition function. We verify this claim in the next section. The proposal of [18] is as follows: the refined vertex corresponding to figure 3 is given by

$$C_{\lambda\mu\nu}(t,q) = \left(\frac{q}{t}\right)^{\frac{\|\mu\|^2+\|\nu\|^2}{2}} t^{\frac{\kappa_\mu}{2}}\, P_{\nu^t}(t^{-\rho};q,t) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho}\, q^{-\nu})\, s_{\mu/\eta}(t^{-\nu^t}\, q^{-\rho}) \tag{2.18}$$

and we glue the "t-edge" and the "q-edge" with weight

$$f_\mu(t,q) = (-1)^{|\mu|}\, t^{n(\mu)}\, q^{-n(\mu^t)} \tag{2.19}$$

The specialization of Macdonald function $P_{\nu^t}(t^{-\rho};q,t)$, $||\mu||^2$, and $n(\mu)$ are defined in appendix A.

The purpose of this article is to confirm that the refined vertex for SU(N) geometry engineers the K-theoretic version of the Nekrasov's partition function. The refined partition functions for the SU(2) and SU(3) geometries and their blow-up were computed in [18]. Hence in the next section, we generalize their discussion to the general SU(N) geometries and their blow-up. As the result, we propose that we shoud modifiy the framing factors in order to engineer the Nekrasov's results.

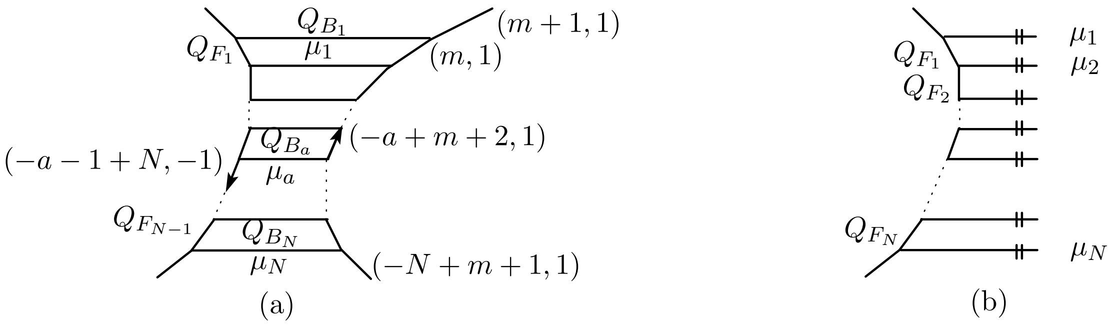

Figure 4: (a)The toric diagram of SU(N) geometry (b)The building block of SU(N) geometry, and refined vertex on this geometry implies $K_{\mu_1\cdots\mu_N}\left(Q_{F,1},\ldots,Q_{F,N-1}\right)$

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3. Refined A-model amplitudes and Nekrasov's partition functions

In this section, we compute the refined partition function for SU(N) geometry via refined topological vertex. The important point of the result in this section is that these refined partition functions are the same as the K-theoretic version of the Nekrasov's partition functions under the little modification of the framing factor. This result verifies the proposal of the refined topological vertex.

3.1 $\mathcal{N}=2$ SU(N) super Yang-Mills

3.1.1 A-model partition function

The toric diagram of SU(N) geometry which give rise to the $\mathcal{N}=2$ SU(N) super Yang-Mills are shown in figure 4(a). The parallel edges corresponding to the base $\mathbb{P}^1$ are the preferred directions of [18]. For fixed N, there are $N+1$ inequivalent geometries ($m=0\cdots N$) which give SU(N) super Yang-Mills. The number m is the Chern-Simons coefficient of the 5-dimensiomal theory in the gauge theory side.

Let us start with the computation of the subdiagram figure 4(b). For the reason which we discuss later, we modify slightly the framing factor proposed in [18] as follows

$$f_\mu(t,q)=(-1)^{|\mu|}\,t^{\frac{||\mu^t||^2}{2}}\,q^{-\frac{||\mu||^2}{2}}=(-1)^{|\mu|}\left(\frac{t}{q}\right)^{\frac{||\mu^t||^2}{2}}q^{-\frac{\kappa_\mu}{2}} \tag{3.1}$$

Using the refined vertex, we can express the subdiagram as

$$\begin{aligned}
K_{\mu_1\cdots\mu_N}\left(Q_{F,1},\ldots,Q_{F,N-1}\right)&=\sum_{\lambda_1\cdots\lambda_{N-1}}\prod_{a=1}^{N}\left(-Q_{F,a}\right)^{||\lambda_a||}f_{\lambda_a}(t,q)\,C_{\lambda^t_{a-1}\,\lambda_a\,\mu_a}(t,q)\\
&=\sum_{\lambda_1\cdots\lambda_{N-1}}\prod_{a=1}^{N}\sum_{\eta_1\cdots\eta_N}Q_{F,a}^{|\lambda_a|}\left(\frac{t}{q}\right)^{\frac{||\lambda^t_a||^2}{2}}q^{-\frac{\kappa_{\lambda_a}}{2}}\left(\frac{q}{t}\right)^{\frac{||\lambda_a||^2+||\mu_a||^2}{2}}\\
&\quad\times t^{\frac{\kappa_{\lambda_a}}{2}}\,P_{\mu^t_a}\left(t^{-\rho};q,t\right)\left(\frac{q}{t}\right)^{\frac{|\eta_a|+|\lambda_{a-1}|-|\lambda_a|}{2}}\\
&\quad\times s_{\lambda_{a-1}/\eta_a}\left(t^{-\rho}\,q^{-\mu_a}\right)s_{\lambda_a/\eta_a}\left(t^{-\mu_a{}^t}\,q^{-\rho}\right)
\end{aligned} \tag{3.2}$$

Notice $\|\mu\|^2 - \|\mu^t\|^2 = \kappa_\mu$ and $\lambda_0 = \lambda_N = \phi$. Simplifing the summation, we get

$$K_{\mu_1 \cdots \mu_N}\left(Q_{F,1}, \ldots, Q_{F,N-1}\right) = \prod_{a=1}^{N}\left[q^{\frac{\|\mu_a\|^2}{2}}\,\tilde{Z}_{\mu_a}(t,q)\right]$$
$$\times \sum_{\lambda_1 \cdots \lambda_{N-1}} \sum_{\eta_1 \cdots \eta_N} \prod_{a=1}^{N} Q_{F,a}^{|\lambda_a|}\left(\frac{q}{t}\right)^{\frac{|\eta_a|}{2}} s_{\lambda_{a-1}/\eta_a}\left(t^{-\rho}\,q^{-\mu_a}\right) s_{\lambda_a/\eta_a}\left(t^{-\mu_a{}^t}\,q^{-\rho}\right)$$

The sum involved in the above subdiagram becomes

$$\sum_{\lambda_1 \cdots \lambda_{N-1}} \sum_{\eta_1 \cdots \eta_N} \prod_{a=1}^{N}\left(\sqrt{\frac{q}{t}}\,Q_{F,a}\right)^{|\lambda_a|} s_{\lambda_{a-1}/\eta_a}\left(t^{-\rho}\,q^{-\mu_a - \frac{1}{2}}\right) s_{\lambda_a/\eta_a}\left(t^{-\mu_a{}^t + \frac{1}{2}}\,q^{-\rho}\right)$$
$$= \sum_{\lambda_1 \cdots \lambda_{N-1}} \sum_{\rho_1 \cdots \rho_{N-1}} \prod_{a=1}^{N-1}\left(\sqrt{\frac{q}{t}}\,Q_{F,a}\right)^{|\lambda_a|} s_{\lambda_a/\rho_{a-1}}\left(t^{-\mu_a{}^t + \frac{1}{2}}\,q^{-\rho}\right) s_{\lambda_a/\rho_a}\left(t^{-\rho}\,q^{-\mu_{a+1} - \frac{1}{2}}\right)$$

Notice taht $\rho_0 = \rho_{N-1} = \phi$. We can take the summation over Young diagrams by Lemma 3.1 of [12], or using the vertex on a strip [27] as we will disscuss in the next subsection. Then we get

$$K_{\mu_1 \cdots \mu_N}\left(Q_{F,a}\right) = \prod_{a=1}^{N}\left[q^{\frac{\|\mu_a\|^2}{2}}\,\tilde{Z}_{\mu_a}(t,q)\right]$$
$$\times \prod_{1 \le a < b \le N} \prod_{i,j=1}^{\infty} \frac{1}{1 - Q_{ab}\,t^{-\mu_{ai}^t + j}\,q^{-\mu_{bj} + i - 1}} \tag{3.3}$$

where $Q_{ab} \equiv \prod_{l=a}^{b-1}\left(\sqrt{\frac{q}{t}}\,Q_{F,l}\right) \equiv \prod_{l=a}^{b-1} \tilde{Q}_{F,l}$.

Let us glue these subdiagrams. The framing factors are given by $n_a = (a - m - 2, -1) \wedge (a - N + 1, 1) = -(N + m - 2a + 1)$ as figure 4. Then, the A-model amplitude is

$$Z^{A-model,\mathrm{SU}(N)}\left(Q_B, Q_{F,a}\right) = \sum_{\mu_1 \cdots \mu_N} \prod_{a=1}^{N}\left[Q_{B,a}^{|\mu_a|}\,f_{\mu_a}(t,q)^{n_a}\right]$$
$$K_{\mu_1 \cdots \mu_N}\left(Q_{F,a}, t, q\right) K_{\mu_N^t \cdots \mu_1^t}\left(Q_{F,a}, q, t\right)$$
$$= Z_{pert.}^{A-model,\mathrm{SU}(N)}\left(Q_{F,a}\right) Z_{\mathrm{inst}}^{A-model,\mathrm{SU}(N)}\left(Q_B, Q_{F,a}\right) \tag{3.4}$$

The perturbative part of the partition function is given by [7]

$$Z_{pert.}^{A-model,\mathrm{SU}(N)}\left(Q_{F,a}\right) \equiv K_{\phi \cdots \phi}\left(Q_{F,a}\right)^2$$

By substituting (3.3) into (3.4), we obtain

$$Z_{inst.}^{A-model,\mathrm{SU}(N)}\left(Q_B, Q_{F,a}\right) = \sum_{\mu_1 \cdots \mu_N} \prod_{a=1}^{N}\left[Q_{B,a}^{|\mu_a|}\,f_{\mu_a}(t,q)^{n_a}\,q^{\frac{\|\mu_a\|^2}{2}}\,t^{\frac{\|\mu_a^t\|^2}{2}}\,\tilde{Z}_{\mu_a}(t,q)\,\tilde{Z}_{\mu_a^t}(q,t)\right]$$
$$\times \prod_{1 \le a < b \le N} \prod_{i,j=1}^{\infty} \frac{1 - Q_{ab}\,t^{j}\,q^{i-1}}{1 - Q_{ab}\,t^{-\mu_{ai}^t + j}\,q^{-\mu_{bj} + i - 1}}\,\frac{1 - Q_{ab}\,t^{j-1}\,q^{i}}{1 - Q_{ab}\,t^{-\mu_{ai}^t + j - 1}\,q^{-\mu_{bj} + i}} \tag{3.5}$$

As we show in the next subsection, the partition function is identical with that of Nekrasov.

3.1.2 Identification with Nekrasov's partition functions

In this subsection, we show that the refined A-model amplitude agrees with the K-theoretic version of the Nekrasov's partition function:

$$Z_{inst.}^{A-model,\mathrm{SU}(N)}\left(Q_B,Q_{F,a}\right)=Z_{inst.}^{Nek,\mathrm{SU}(N)}\left(\hat{Q},Q_{ab}\right) \tag{3.6}$$

Recall that the K-theoretic version of the Nekrasov's partition functions with a Chern-Simons term is given by [28, 29]

$$Z_{inst.}^{Nek.\,\mathrm{SU}(N),m}(\hat{Q},Q_{ab})=\sum_{\vec{\mu}}\frac{\hat{Q}^{|\vec{\mu}|}}{\prod_{a,b}N_{ab}^{\vec{\mu}}(t,q,Q_{ab})}\left(\frac{q}{t}\right)^{\frac{N}{2}|\vec{\mu}|}\prod_{a=1}^{N}e_a^{m|\mu_a|}\,t^{-m\frac{||\mu_a^t||^2}{2}}\,q^{m\frac{\|\mu_a\|^2}{2}} \tag{3.7}$$

Note that $Q_{ab}=e_a\,e_b^{-1}$.

First, let us rewrite the character part $\prod N_{ab}^{\vec{\mu}}$. The identity

$$\sum_{i,j=1}^{\infty}q^{\mu_i-j+1}\,t^{\nu_j-i}=\sum_{i,j=1}^{\infty}q^{-\nu_j^t+i}\,t^{-\mu_i^t+j-1} \tag{3.8}$$

follows from $(t-1)\sum_{i=1}^{\infty}q^{\mu_i}\,t^{-i}=\left(q^{-1}-1\right)\sum_{i=1}^{\infty}t^{-\mu_i^t}\,q^i$ for $t,q\neq 1$ [16]. It is easy to prove the following formula using (3.8) (take the logarithm of the equation(3.9))

$$\prod_{i,j=1}^{\infty}\left(1-Q\,t^{-\mu_j^t+i}\,q^{-\nu_i+j-1}\right)=\prod_{i,j=1}^{\infty}\left(1-Q\,q^{\mu_i-j}\,t^{\nu_j^t-i+1}\right) \tag{3.9}$$

The character part of the Nekrasov's patririon function is given by

$$\begin{aligned}\frac{1}{N_{12}^{\vec{\mu}}\,(t,q,Q)}&\equiv\prod_{(i,j)\in\mu}\frac{1}{1-Q\,t^{\nu_j^t-i}\,q^{\mu_i-j+1}}\prod_{(i,j)\in\nu}\frac{1}{1-Q\,t^{-\mu_j^t+i-1}\,q^{-\nu_i+j}}\\&=\prod_{i,j=1}^{\infty}\frac{1-Q\,t^{j-1}\,q^i}{1-Q\,t^{-\mu_j^t+i-1}\,q^{-\nu_i+j}}\end{aligned} \tag{3.10}$$

where $\mu_1=\mu,\mu_2=\nu,Q_{12}=Q$. By using (3.9), we have

$$\begin{aligned}\prod_{i,j=1}^{\infty}\frac{1-Q\,t^j\,q^{i-1}}{1-Q\,t^{-\mu_j^t+i}\,q^{-\nu_i+j-1}}&=\prod_{i,j=1}^{\infty}\frac{1-Q\,t^j\,q^{i-1}}{1-Q\,q^{\mu_i-j}\,t^{\nu_j^t-i+1}}\\&=\prod_{(i,j)\in\nu}\frac{1}{1-Q\,t^{-\mu_j^t+i}\,q^{-\nu_i+j-1}}\prod_{(i,j)\in\mu}\frac{1}{1-Q\,t^{\nu_j^t-i+1}\,q^{\mu_i-j}}\\&=(-Q)^{-|\mu|-|\nu|}\,t^{\sum_{(i,j)\in\nu}(\mu_j^t-i)-\sum_{(i,j)\in\mu}(\nu_j^t-i+1)}\\&\qquad q^{\sum_{(i,j)\in\nu}(\nu_i-j+1)-\sum_{(i,j)\in\mu}(\mu_i-j)}\times\frac{1}{N_{21}^{\vec{\mu}}\,(t,q,Q^{-1})}\end{aligned} \tag{3.11}$$

The factors appear in the above equation become

$$\sum_{(i,j)\in\nu}\mu_j^t=\sum_{j=1}^{\nu_1}\sum_{i=1}^{\nu_j^t}\mu_j^t=\sum_{j=1}^{\min(\mu_1,\nu_1)}\mu_j^t\,\nu_j^t=\sum_{(i,j)\in\mu}\nu_j^t$$

$$\sum_{(i,j)\in\mu}(\mu_i-j)=\sum_{i=1}^{d(\mu)}[(\mu_i-1)+\cdots+(\mu_i-\mu_i)]=\frac{\|\mu\|^2}{2}-\frac{|\mu|}{2}$$

Hence we obtain

$$\frac{1}{N_{12}^{\vec{\mu}}(t,q,Q)\,N_{21}^{\vec{\mu}}(t,q,Q^{-1})}=(-Q)^{|\mu|+|\nu|}\left(\frac{q}{t}\right)^{-\frac{|\mu|}{2}-\frac{|\nu|}{2}+\frac{||\mu^t||^2}{2}-\frac{||\nu^t||^2}{2}}q^{\frac{\kappa_\mu}{2}-\frac{\kappa_\nu}{2}}\tag{3.12}$$
$$\times\prod_{i,j=1}^{\infty}\frac{1-Q_{12}\,t^{i-1}\,q^j}{1-Q_{12}\,t^{-\mu_j^t+i-1}\,q^{-\nu_i+j}}\frac{1-Q_{12}\,t^i\,q^{j-1}}{1-Q_{12}\,t^{-\mu_j^t+i}\,q^{-\nu_i+j-1}}$$

It is easy to show

$$\frac{1}{N_{aa}^{\vec{\mu}}(t,q,Q_{aa}=1)}=(-1)^{|\mu_a|}\left(\frac{t}{q}\right)^{\frac{|\mu_a|}{2}}t^{\frac{||\mu_a^t||^2}{2}}q^{\frac{\|\mu_a\|^2}{2}}\tilde{Z}_{\mu_a}(t,q)\,\tilde{Z}_{\mu_a^t}(q,t)\tag{3.13}$$

By combining above identities, we can rewrite the Nekrasov's partition function as follows

$$\sum_{\vec{\mu}}\frac{\hat{Q}^{|\vec{\mu}|}}{\prod_{a<b}N_{ab}^{\vec{\mu}}(t,q,Q_{ab})}\left(\frac{q}{t}\right)^{\frac{N}{2}|\vec{\mu}|}\prod_{a=1}^{N}e_a^{m|\mu_a|}\,t^{-m\frac{||\mu_a^t||^2}{2}}\,q^{m\frac{\|\mu_a\|^2}{2}}$$
$$=\sum_{\vec{\mu}}(-1)^{N|\vec{\mu}|}\hat{Q}^{|\vec{\mu}|}\prod_{a<b}\left[Q_{ab}^{|\mu_a|+|\mu_b|}\right]\prod_{a=1}^{N}[e_a^{m|\mu_a|}\left(\frac{q}{t}\right)^{(N+m-2a+1)\frac{||\mu_a^t||^2}{2}}q^{(N+m-2a+1)\frac{\kappa_{\mu_a}}{2}}$$
$$\times\,t^{\frac{||\mu_a^t||^2}{2}}\,q^{\frac{\|\mu_a\|^2}{2}}\,\tilde{Z}_{\mu_a}(t,q)\,\tilde{Z}_{\mu_a^t}(q,t)]$$
$$\times\prod_{a<b}\prod_{i,j=1}^{\infty}\frac{1-Q_{ab}\,t^{i-1}\,q^j}{1-Q_{ab}\,t^{-\mu_{aj}^t+i-1}\,q^{-\mu_{bi}+j}}\frac{1-Q_{ab}\,t^i\,q^{j-1}}{1-Q_{ab}\,t^{-\mu_{aj}^t+i}\,q^{-\mu_{bi}+j-1}}\tag{3.14}$$

Next, let us rewrite the remainder $\hat{Q}^{|\vec{\mu}|}\prod_{a<b}Q_{ab}^{|\mu_a|+|\mu_b|}\prod_{a=1}^{N}e_a^{m|\mu_a|}$. We shall rewrite it in terms of the Kähler parameters of the base and the fiber $\mathbb{P}^1$'s by showing the following identity

$$C\,{Q_B}^{|\vec{\mu}|}\prod_{a<b}Q_{ab}^{|\mu_a|+|\mu_b|}\prod_{a=1}^{N}e_a^{m|\mu_a|}=\prod_{a=1}^{N}Q_{B,a}^{|\mu_a|}\tag{3.15}$$

We prove this identity in the case of N =odd and m =even for example. It is easy to generarize this proof. First we use the results of [9], that is, $Q_{B,a}$ are given by the base and the fiber Kähler parameters and they satisfy

$$\prod_{i=1}^{N}Q_{B,a}^{|\mu_a|}={Q_B}^{|\vec{\mu}|}\prod_{a=1}^{\left[\frac{N+m-1}{2}\right]}\tilde{Q}_{F,a}^{(N+m-2a)(|\mu_1|+\cdots+|\mu_a|)}\prod_{a=\left[\frac{N+m}{2}+1\right]}^{N-1}\tilde{Q}_{F,a}^{-(N+m-2a)(|\mu_{a+1}|+\cdots+|\mu_N|)}\tag{3.16}$$

Here we modify $Q_{F,a}$ to $\tilde{Q}_{F,a}$ in the case of the refined partition function.

On the one hand we can obtain the following identity after some algebra

$$\begin{aligned}
\prod_{a<b} Q_{ab}^{|\mu_a|+|\mu_b|} &= \left(\prod_{a=1}^{\frac{N-1}{2}} \tilde{Q}_{F,a}^{a} \prod_{a=\frac{N+1}{2}}^{N-1} \tilde{Q}_{F,a}^{N-a}\right)^{|\vec{\mu}|} \prod_{a=1}^{\frac{N-1}{2}} \left(\prod_{b=a}^{\frac{N-1}{2}} \tilde{Q}_{F,b}^{(N-2b)}\right)^{|\mu_a|} \\
&\quad \prod_{a=\frac{N+1}{2}+1}^{N-1} \left(\prod_{b=\frac{N+1}{2}}^{a-1} \tilde{Q}_{F,b}^{(2b-N)}\right)^{|\mu_a|} \\
&= \left(\prod_{a=1}^{\frac{N-1}{2}} \tilde{Q}_{F,a}^{a} \prod_{a=\frac{N+1}{2}}^{N-1} \tilde{Q}_{F,a}^{N-a}\right)^{|\vec{\mu}|} \prod_{a=1}^{\frac{N-1}{2}} \tilde{Q}_{F,a}^{(N-2a)(|\mu_1|+\cdots+|\mu_a|)} \\
&\quad \prod_{a=\frac{N+1}{2}}^{N-1} \tilde{Q}_{F,a}^{(2a-N)(|\mu_{a+1}|+\cdots+|\mu_N|)} \qquad (3.17)
\end{aligned}$$

Here we use $Q_{ab} = \prod_{l=a}^{b-1} \tilde{Q}_{F,l}$. Using $\tilde{Q}_{F,a} = e_a\, {e_{a+1}}^{-1}$, we can also show

$$\begin{aligned}
&\prod_{a=1}^{\frac{N-1}{2}} \tilde{Q}_{F,a}^{m(|\mu_1|+\cdots+|\mu_a|)} \prod_{a=\frac{N+1}{2}}^{N-1} \tilde{Q}_{F,a}^{-m(|\mu_{a+1}|+\cdots+|\mu_N|)} \\
&\qquad \times \prod_{a=\frac{N+1}{2}}^{\frac{N+m-1}{2}} \tilde{Q}_{F,a}^{(N+m-2a)(|\mu_1|+\cdots+|\mu_a|)} \prod_{a=\frac{N+1}{2}}^{\frac{N+m-1}{2}} \tilde{Q}_{F,a}^{(-2a+N+m)(|\mu_{a+1}|+\cdots+|\mu_N|)} \\
&= \left(\prod_{a=1}^{\frac{N-1}{2}} \prod_{b=a}^{\frac{N-1}{2}} \tilde{Q}_{F,b}^{|\mu_a|} \prod_{a=\frac{N+1}{2}+1}^{N-1} \prod_{b=\frac{N+1}{2}}^{a-1} \tilde{Q}_{F,b}^{-|\mu_a|}\right)^{m} \left(\prod_{a=\frac{N+1}{2}}^{\frac{N+m-1}{2}} \tilde{Q}_{F,a}^{(N+m-2a)}\right)^{(|\mu_1|+\cdots+|\mu_N|)} \\
&= \left(\left(e_{\frac{N+1}{2}}\right)^{-m} \prod_{a=\frac{N+1}{2}}^{\frac{N+m-1}{2}} \tilde{Q}_{F,a}^{(N+m-2a)}\right)^{|\vec{\mu}|} \prod_{a=1}^{N-1} e_a^{m|\mu_a|} \qquad (3.18)
\end{aligned}$$

$$C = \frac{\prod_{a=\frac{N+1}{2}}^{\frac{N+m-1}{2}} Q_{F,a}^{N+m-2a}}{(e_{\frac{N+1}{2}})^m \prod_{a=1}^{\frac{N-1}{2}} Q_{F,a}^{a} \prod_{a=\frac{N+1}{2}}^{N-1} Q_{F,a}^{N-a}} \qquad (3.19)$$

Finally, (3.14) (3.15) imply the following equality

$$Z_{inst.}^{A-model,\mathrm{SU}(N)}\left(Q_B, Q_{F,a}\right) = Z_{inst.}^{Nek,\mathrm{SU}(N)}\left(\hat{Q}, Q_{ab}\right) \qquad (3.20)$$

where $\hat{Q} = (-1)^N C(e_a, t, q, m)\, Q_B$.

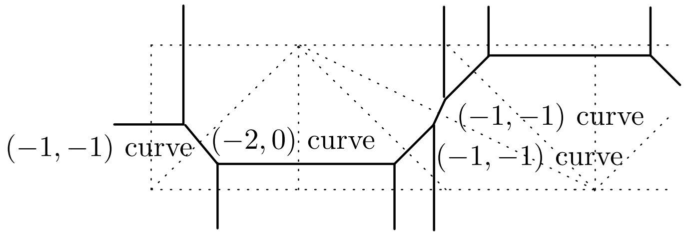

Figure 5: A toric diagram of a strip geometry which is obtained from triangulation of a strip toric data

3.2 Modification of the framing factor

Recall that we adopt the modified framing factor

$$f_\mu(t,q) = (-1)^{|\mu|}\, t^{\frac{||\mu^t||^2}{2}}\, q^{-\frac{||\mu||^2}{2}} \tag{3.21}$$

in the calculation of this section. If we use the framing factor without our modification, we get the additional factor

$$\prod_{a=1}^{N} \left(\frac{t}{q}\right)^{-\frac{n_a|\mu_a|}{2}} \tag{3.22}$$

in the summation of the refined patririon function. They cannot be absorbed into the $\hat{Q}$ and break the equivalence (3.20). Hence we need the modification of the framing factor (3.1).

n_a are small integers for the SU(2) and SU(3) geometries, so the partition functions are insensitive to the factor (3.22) in [18]. On the one hand, the mismatch (3.22) is obvious for SU(N) geometry with general Chern-Simon term m.

3.3 Adding matters and strip geometries

By blowing up the SU(N) geometries, we can add matters to the Nekrasov's instanton calculation via the geometric engineering. The Kähler parameters of the blown up $\mathbb{P}^1$'s give rise to the mass parameters of the matters. These geometries is obtained by gluing strip geometries. A strip geometry is a toric Calabi-Yau that contains a chain of $\mathbb{P}^1$'s. Each $\mathbb{P}^1$ locally forms a $(-1,-1)$ curve $O(-1)\oplus O(-1) \to \mathbb{P}^1$ or $(-2,0)$ curve $O(-2)\oplus O(0) \to \mathbb{P}^1$ as figure 5. Following [27], we take the chain of $(-1,-1)$ curves figure 6 for example. Gluing these strip geometries, we get the toric Calabi-Yau that engineers $\mathcal{N}= 2$ SU(N) gauge theory with $N_f = 2N$ [27]. The refined vertex on the strip geometry figure 6(a) yields

$$\begin{aligned} K^{\alpha_1\,\alpha_2\ldots}_{\beta_1\,\beta_2\cdots} = &\sum_{\{\mu_a\},\{\nu_a\}} \left(-Q_{M,1}\right)^{|\mu_1|} \left(-Q_{F,2}\right)^{|\nu_2|} \left(-Q_{M,2}\right)^{|\mu_2|} \cdots \\ &\times C_{\nu_1^t\,\mu_1\,\alpha_1}\, C_{\nu_2\,\mu_1^t\,\beta_1}\, C_{\nu_2^t\,\mu_2\,\alpha_2}\, C_{\nu_3\,\mu_2^t\,\beta_2} \times \cdots \end{aligned} \tag{3.23}$$

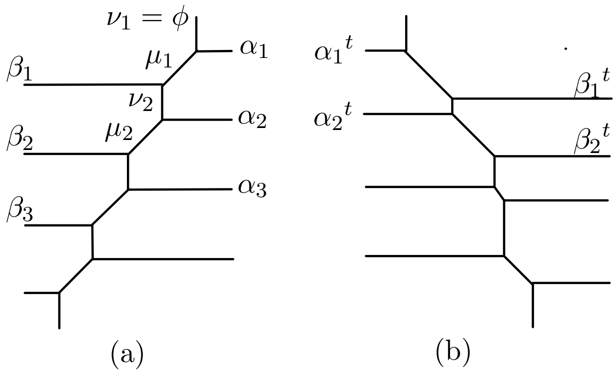

Figure 6: The building blocks of the toric Calabi-Yau that engineers SU(N) gauge theory with $N_f = 2N$

$$
\begin{aligned}
&= \prod_a \left[q^{\frac{\|\alpha_a\|^2}{2}}\, t^{\frac{\|\beta_a\|^2}{2}}\, \tilde{Z}_{\alpha_a}(t,q)\, \tilde{Z}_{\beta_a}(q,t) \right] \\
&\quad \times \sum_{\substack{\{\mu_a\},\{\nu_a\} \\ \{\rho_a\},\{\sigma_a\}}} \prod_a (-Q_{M,a})^{|\mu_a|}\, (-Q_{F,a})^{|\nu_a|} \left(\frac{q}{t}\right)^{\frac{\|\mu_a\|^2}{2}} t^{\frac{\kappa_{\mu_a}}{2}} \left(\frac{t}{q}\right)^{\frac{||\mu_a^t||^2}{2}} q^{-\frac{\kappa_{\mu_a}}{2}} \\
&\quad \times \left(\frac{q}{t}\right)^{\frac{|\rho_a|+|\nu_a|-|\mu_a|}{2}} \left(\frac{t}{q}\right)^{\frac{|\sigma_a|+|\nu_{a+1}|-|\mu_a|}{2}} \\
&\quad \times s_{\nu_a/\rho_a}\left(t^{-\rho}\, q^{-\alpha_a}\right) s_{\mu_a/\rho_a}\left(t^{-\alpha_a^t}\, q^{-\rho}\right) s_{\nu^t_{a+1}/\sigma_a}\left(q^{-\rho}\, t^{-\beta_a}\right) s_{\mu^t_a/\sigma_a}\left(q^{-\beta_a^t}\, t^{-\rho}\right)
\end{aligned}
$$

where $\nu_1 = \nu_{N+1} = \phi$. They involve the following sum

$$
\begin{aligned}
&\sum_{\substack{\{\mu_a\},\{\nu_a\} \\ \{\rho_a\},\{\sigma_a\}}} \prod_a (-Q_{M,a})^{|\mu_a|}\, (-Q_{F,a})^{|\nu_a|} \\
&\times s_{\nu_a/\rho_a}\left(t^{-\rho+\frac{1}{2}}\, q^{-\alpha_a}\right) s_{\mu_a/\rho_a}\left(t^{-\alpha_a^t}\, q^{-\rho-\frac{1}{2}}\right) s_{\nu^t_{a+1}/\sigma_a}\left(q^{-\rho}\, t^{-\beta_a-\frac{1}{2}}\right) s_{\mu^t_a/\sigma_a}\left(q^{-\beta_a^t+\frac{1}{2}}\, t^{-\rho}\right)
\end{aligned}
$$

Using the method of Iqbal-KashaniPoor [27], we can take the summation. The only difference from the result of [27] is the arguments of Schur functions. Bewaring the difference, we get the sum as follows

$$
\begin{aligned}
&\prod_{1\le a\le b\le N} \left[t^{-\alpha_a^t}\, q^{-\rho-\frac{1}{2}}, -Q_{\alpha_a\beta_b}\, q^{-\beta_b^t+\frac{1}{2}}\, t^{-\rho}\right] \prod_{1\le a<b\le N} \left[q^{-\rho}\, t^{-\beta_a-\frac{1}{2}}, -Q_{\beta_a\alpha_b}\, t^{-\rho+\frac{1}{2}}\, q^{-\alpha_b}\right] \\
&\times \prod_{1\le a<b\le N} \left\{t^{-\alpha_a^t}\, q^{-\rho-\frac{1}{2}}, Q_{\alpha_a\alpha_b}\, t^{-\rho+\frac{1}{2}}\, q^{-\alpha_b}\right\} \left\{q^{-\rho}\, t^{-\beta_a-\frac{1}{2}}, Q_{\beta_a\beta_b}\, t^{-\rho}\, q^{-\beta_b^t+\frac{1}{2}}\right\} \qquad (3.24)
\end{aligned}
$$

We provide the direct proof in appendix B. Here Kähler parameters are given by

$$
\begin{aligned}
Q_{\alpha_a\beta_b} &= Q_{M,a}\, Q_{F,a+1} \cdots Q_{M,b-1}\, Q_{F,b}\, Q_{M,b} = Q_{a,b}\, Q_{M,b} \\
Q_{\beta_a\alpha_b} &= Q_{F,a+1} \cdots Q_{M,b-1}\, Q_{F,b} = Q^{-1}_{M,a}\, Q_{a,b} \\
Q_{\alpha_a\alpha_b} &= Q_{M,a}\, Q_{F,a+1} \cdots Q_{M,b-1}\, Q_{F,b} = Q_{a,b} \\
Q_{\beta_a\beta_b} &= Q_{F,a+1} \cdots Q_{M,b-1}\, Q_{F,b}\, Q_{M,b} = Q^{-1}_{M,a}\, Q_{a,b}\, Q_{M,b}
\end{aligned}
$$

and we introduce

$$[x,y] \equiv \prod_{i,j=1}^{\infty} (1+x_i\,y_j), \{x,y\} \equiv \prod_{i,j=1}^{\infty} (1-x_i\,y_j)^{-1} \tag{3.25}$$

Then we obtain the following expression

$$\begin{aligned}
K^{\alpha_1\,\alpha_2\cdots}_{\beta_1\,\beta_2\cdots} = & \prod_a \left[q^{\frac{\|\alpha_a\|^2}{2}}\, t^{\frac{\|\beta_a\|^2}{2}}\, \tilde{Z}_{\alpha_a}(t,q)\, \tilde{Z}_{\beta_a}(q,t) \right] \\
& \times \prod_{i,j=1}^{\infty} \prod_{1\le a\le b\le N} \left(1 - Q_{\alpha_a\,\beta_b}\, t^{-\,\alpha^t_{a,i}\,+j-\frac{1}{2}}\, q^{-\,\beta^t_{b,j}\,+i-\frac{1}{2}}\right) \\
& \prod_{1\le a<b\le N} \left(1 - Q_{\beta_a\,\alpha_b}\, t^{-\,\beta_{a,i}\,+j-\frac{1}{2}}\, q^{-\,\alpha_{b,j}\,+i-\frac{1}{2}}\right) \\
& \times \prod_{1\le a<b\le N} \left(1 - Q_{\alpha_a\,\alpha_b}\, t^{-\,\alpha^t_{a,i}\,+j}\, q^{-\,\alpha_{b,j}\,+i-1}\right)^{-1} \left(1 - Q_{\beta_a\,\beta_b}\, t^{-\,\beta_{a,i}\,+j-1}\, q^{-\,\beta^t_{b,j}\,+i}\right)^{-1}
\end{aligned} \tag{3.26}$$

The amplitude for the pair of this strip geomerty figure 6(b) is given by

$$\begin{aligned}
\tilde{K}^{\beta^t_1\,\beta^t_2\cdots}_{\alpha^t_1\,\alpha^t_2\cdots} = & \prod_a \left[t^{\frac{||\alpha^t_a||^2}{2}}\, q^{\frac{||\beta^t_a||^2}{2}}\, \tilde{Z}_{\alpha^t_a}(q,t)\, \tilde{Z}_{\beta^t_a}(t,q) \right] \\
& \times \prod_{i,j=1}^{\infty} \prod_{1\le a\le b\le N} \left(1 - Q'_{\alpha_a\,\beta_b}\, t^{-\,\alpha^t_{a,i}\,+j-\frac{1}{2}}\, q^{-\,\beta^t_{b,j}\,+i-\frac{1}{2}}\right) \\
& \prod_{1\le a<b\le N} \left(1 - Q'_{\beta_a\,\alpha_b}\, t^{-\,\beta_{a,i}\,+j-\frac{1}{2}}\, q^{-\,\alpha_{b,j}\,+i-\frac{1}{2}}\right) \\
& \times \prod_{1\le a<b\le N} \left(1 - Q'_{\alpha_a\,\alpha_b}\, t^{-\,\alpha^t_{a,i}\,+j-1}\, q^{-\,\alpha_{b,j}\,+i}\right)^{-1} \left(1 - Q'_{\beta_a\,\beta_b}\, t^{-\,\beta_{a,i}\,+j}\, q^{-\,\beta^t_{b,j}\,+i-1}\right)^{-1}
\end{aligned} \tag{3.27}$$

Gluing them, we get the Nekrasov's partition functioin for $\mathcal{N}= 2$ SU(N) gauge theory with $N_f = 2N$

$$Z = \sum_{\alpha_1\,\alpha_2\cdots} \prod_{a=1}^{N} \left(f_{\alpha_a}(t,q)\, Q_B{}^{|\alpha_a|} \right) K^{\alpha_1\,\alpha_2\cdots}_{\phi\phi\cdots\cdots}(Q_{ab}, Q_{M,a})\, \tilde{K}^{\phi\phi\cdots}_{\alpha^t_1\,\alpha^t_2\cdots}(Q_{ab}, Q'_{M,a})$$

It is not so hard to generarize the above caluculation of the refined vertex for another strip geometries which contain $(-1,-1)$ curves and $(-2,0)$ curves. Then we can engineer the Nekrasov's partition functioins for various $\mathcal{N}= 2$ SU(N) quiver gauge theories with matters by gluing these amplituses.

4. Conclusion

In this paper, we have applied refined topological vertex for SU(N) geometries and reproduced the K-theoretic version of the Nekrasov's partition functions. From this results

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we can adopt refined topological vertex as a 2-parameter extension of topological A-model under the modification of the framing factor. As we discussed in this paper, the necessity of the modification become clear in the case of SU(N) geometry. We have also discussed a refined vertex on a strip geometry. Many of the nice properties obtained in [27] are maintained in the case of refined vertex. The important point is that refined vertex on strip reduces to a summation of Schur functions which is essentially discussed in [27]. Hence Schur functions of the partition functions can be summed up as in the case of the topological vertex on strips.

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A. Young diagrams and Schur functions

Young diagrams. The Young diagrams is defined as a sequence of decreasing non-negative integers

$$\mu = \{\mu_i \in \mathbb{Z}_{\geq 0} \,|\, \mu_1 \geq \mu_2 \geq \cdots\} \tag{A.1}$$

The transpose of μ is defined as follows

$$\mu^t = \left\{\mu^t_j \in Z_{\geq 0} \,|\, \mu^t_j = \#\left\{i |\, \mu_i \geq j\right\}\right\} \tag{A.2}$$

The size and the norm of the partition is denoted as

$$|\mu| = \sum_{i=1}^{d(\mu)} \mu_i, \quad \|\mu\|^2 = \sum_{i=1}^{d(\mu)} \mu_i^2 \tag{A.3}$$

For $(i,j) \in \mu$, we define the following quantities,

$$\begin{aligned} a_\mu(i,j) &= \mu_i - j, l_\mu(i,j) = \mu^t_j - i \\ a'_\mu(i,j) &= j-1, l'_\mu(i,j) = i-1 \end{aligned}$$

We introduce the hook length of the Young diagram

$$h_\mu(i,j) = \mu_i - j + \mu^t_j - i + 1$$

It is also useful to define the following quantities

$$n(\mu) = \sum_{i=1}^{d(\mu)} (i-1)\,\mu_i \;\; \kappa_\mu = \sum_{(i,j)\in\mu} (j-i)$$

It is easy to show that they satisfy the following identities

$$n(\mu) = \frac{1}{2}\sum_{j=1}^{\mu_1} \mu^t_j(\mu^t_j - 1) = \sum_{s\in\mu} l'_\mu(s) = \sum_{s\in\mu} l_\mu(s) \tag{A.4}$$

$$n(\mu^t) = \frac{1}{2}\sum_{i=1}^{d(\mu)} \mu_i(\mu_i - 1) = \sum_{s\in\mu} a'_\mu(s) = \sum_{s\in\mu} a_\mu(s) \tag{A.5}$$

$$\kappa_\mu = 2(n(\mu^t) - n(\mu)) = \|\mu\|^2 - \left\|\mu^t\right\|^2 \tag{A.6}$$

$$\sum_{s\in\mu} h_\mu(s) = n(\mu) + n(\mu^t) + |\mu| \tag{A.7}$$

Schur functions. The Schur functions for N variables $(x_1, \ldots, x_N)$ are defined by the determinant formula

$$s_\mu(x_1, \ldots, x_N) = \frac{\det_{i,j=1,\ldots N}\left({x_i}^{\mu_j + N - j}\right)}{\det_{i,j=1,\ldots N}\left({x_i}^{N-j}\right)} \tag{A.8}$$

From the definition, the Schur functions are symmetric under the permutation of the variables. Moreover it is known that they form an orthogonal basis of the symmetric polynomials. We can also define the skew Schur functions by

$$s_{\mu/\nu}(x) = \sum_\rho c^\mu_{\nu\rho}\, s_\rho(x) \tag{A.9}$$

Here we introduce the Richardson-Littlewood coefficients $c^\rho_{\mu\nu}$

$$s_\mu(x)\, s_\nu(x) = \sum_\rho c^\rho_{\mu\nu}\, s_\rho(x) \tag{A.10}$$

We have the product expression for the Schur function of the variables $\{q^\rho\} = \{q^{-i+\frac{1}{2}}\}_{i=1,2,\ldots}$ [30]

$$s_\mu(q^{-\rho}) \equiv s_\mu(q^{\frac{1}{2}}, q^{\frac{3}{2}}, \ldots) = q^{\frac{\left\|\mu^t\right\|^2}{2}}\, \tilde{Z}_\mu(q) \tag{A.11}$$

where

$$\tilde{Z}_\mu(q) = \prod_{s\in\mu}\left(1 - q^{h_\mu(s)}\right)^{-1} \tag{A.12}$$

Using this formula, we obtain

$$s_\mu(q^\rho) = q^{\frac{\kappa_\mu}{2}}\, s_{\mu^t}(q^\rho) = (-1)^{|\mu|}\, s_{\mu^t}(q^{-\rho}) \tag{A.13}$$

Let us introduce the 2-parameter extension of $\tilde{Z}_\mu(q)$ by

$$P_\mu(t^{-\rho}; q, t) = t^{\frac{\left\|\mu^t\right\|^2}{2}}\, \tilde{Z}_{\mu^t}(t, q) \tag{A.14}$$

$$\tilde{Z}_\mu(t, q) = \prod_{s\in\mu}\left(1 - t^{a_\mu(s)+1}\, q^{l_\mu(s)}\right)^{-1} \tag{A.15}$$

It appears in the refinement of topological vertex:

$$C_{\phi\phi\mu}(q) = q^{\frac{\|\mu\|^2}{2}}\, \tilde{Z}_\mu(q) \to C_{\phi\phi\mu}(t, q) = q^{\frac{\|\mu\|^2}{2}}\, \tilde{Z}_\mu(t, q) \tag{A.16}$$

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In summing the Schur functions, we use the following identities

$$\sum_{\mu} s_{\mu}(x)\, s_{\mu}(y) \;=\; \prod_{i,j} (1 - x_i\, y_j)^{-1} \tag{A.17}$$

$$\sum_{\mu} s_{\mu^t}(x)\, s_{\mu}(y) \;=\; \prod_{i,j} (1 + x_i\, y_j) \tag{A.18}$$

$$\sum_{\mu} s_{\mu/\rho}(x)\, s_{\mu/\sigma}(y) \;=\; \prod_{i,j} (1 - x_i\, y_j)^{-1} \sum_{\nu} s_{\rho/\nu}(y)\, s_{\sigma/\nu}(x) \tag{A.19}$$

$$\sum_{\mu} s_{\mu^t/\rho}(x)\, s_{\mu/\sigma}(y) \;=\; \prod_{i,j} (1 + x_i\, y_j) \sum_{\nu} s_{\rho/\nu^t}(y)\, s_{\sigma^t/\nu^t}(x) \tag{A.20}$$

$$s_{\mu}(Qx) \;=\; Q^{|\mu|}\, s_{\mu}(x) \tag{A.21}$$

$$s_{\mu/\nu}(Qx) \;=\; Q^{|\mu|-|\nu|}\, s_{\mu/\nu}(x) \tag{A.22}$$

They are important identities which we use throughout the paper.

B. Proof of formula

In this appendix, we prove the following identity for section 3.

$$\begin{aligned}
&\prod_{a=1}^{N} {Q_{M,a}}^{|\mu_a|} \prod_{a=2}^{N} {Q_{F,a}}^{|\nu_a|} \sum_{\substack{\{\mu_i\},\ \{\nu_i\}\\ \{\rho_i\},\ \{\sigma_i\}}} \prod_{a=1}^{N} s_{\nu_a\,/\,\rho_a}\left(w^{(a)}\right) s_{\mu_a\,/\,\rho_a}\left(x^{(a)}\right) s_{{\mu_a}^t\,/\,\sigma_{a+1}} \\
&\qquad\qquad\qquad\qquad\qquad \left(y^{(a)}\right) s_{{\nu_{a+1}}^t\,/\,\sigma_{a+1}}\left(z^{(a+1)}\right) \\
&= \prod_{1\leq a\leq b\leq N} \left[x^{(a)}, Q_{\alpha_a\,\beta_b}\, y^{(b)}\right] \prod_{1\leq a<b\leq N} \left[z^{(a+1)}, Q_{\beta_a\,\alpha_b}\, w^{(b)}\right] \\
&\qquad\qquad\qquad\qquad\qquad \left\{x^{(a)}, Q_{\alpha_a\,\alpha_b}\, w^{(b)}\right\} \left\{z^{(a+1)}, Q_{\beta_a\,\beta_b}\, y^{(b)}\right\}
\end{aligned} \tag{B.1}$$

where we take the sum over the Young diagrams $\mu_1 \cdots \mu_N$, $\nu_2 \cdots \nu_N$, $\rho_2 \cdots \rho_N$, and $\sigma_2 \cdots \sigma_N$. Notice that we denote $\rho_1 = \sigma_{N+1} = \phi$. in the formula.

Let us show the identity. The first line of this equation becomes

$$\begin{aligned}
&\sum_{\substack{\rho_2\cdots\rho_N\\ \sigma_2\cdots\sigma_N}} \prod_{a=2}^{N} {Q_{M,a}}^{|\rho_a|}\, {Q_{F,a}}^{|\sigma_a|} \\
&\qquad \times \sum_{\substack{\mu_1\cdots\mu_N\\ \nu_2\cdots\nu_N}} \prod_{a=1}^{N} s_{\mu_a\,/\,\rho_a}\left(Q_{M,a}\, x^{(a)}\right) s_{{\mu_a}^t\,/\,\sigma_{a+1}}\left(y^{(a)}\right) s_{{\nu_{a+1}}^t\,/\,\sigma_{a+1}} \\
&\qquad\qquad\qquad \left(Q_{F,a+1}\, z^{(a+1)}\right) s_{\nu_{a+1}\,/\,\rho_{a+1}}\left(w^{(a+1)}\right) \\
&= \prod_{a=1}^{N} \left[x^{(a)}, Q_{M,a}\, y^{(a)}\right] \prod_{a=2}^{N} \left[z^{(a+1)}, Q_{F,a+1}\, w^{(a+1)}\right] \sum_{\substack{\alpha_2\cdots\alpha_{N-1}\\ \beta_2\cdots\beta_N}} \sum_{\substack{\rho_2\cdots\rho_N\\ \sigma_2\cdots\sigma_N}} \prod_{a=2}^{N} {Q_{M,a}}^{|\alpha_a|}\, {Q_{F,a}}^{|\beta_a|} \\
&\quad \times \prod_{a=1}^{N} s_{{\sigma_{a+1}}^t\,/\,\alpha_a}\left(Q_{M,a}\, x^{(a)}\right) s_{{\rho_a}^t\,/\,\alpha_a}\left(Q_{M,a}\, y^{(a)}\right) s_{{\rho_{a+1}}^t\,/\,{\beta_{a+1}}^t}
\end{aligned}$$

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$$
\begin{aligned}
&\left(Q_{F,a+1}\, z^{(a+1)}\right) s_{\sigma_{a+1}{}^{t}\,/\,\beta_{a+1}} \left(Q_{F,a+1}\, w^{(a+1)}\right) \\
&= \prod_{a=1}^{N} \left[x^{(a)}, Q_{M,a}\, y^{(a)}\right] \prod_{a=2}^{N-1} \left[z^{(a)}, Q_{F,a}\, w^{(a)}\right] \prod_{a=1}^{N} \left\{x^{(a)}, Q_{M,a}\, Q_{F,a+1}\, w^{(a+1)}\right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left\{z^{(a+1)}, Q_{F,a+1}\, Q_{M,a+1}\, y^{(a+1)}\right\} \\
&\qquad\qquad \times \sum_{\substack{\beta_1\cdots\beta_{N-1}\\ \alpha_2\cdots\alpha_{N-1}}} \sum_{\substack{\gamma_2\cdots\gamma_{N-1}\\ \delta_2\cdots\delta_{N-1}}} \prod_{a=1}^{N-1} Q_{F,a+1}{}^{|\beta_a|} \prod_{a=2}^{N-1} Q_{M,a}{}^{|\alpha_a|} \\
&\quad \times \prod_{a=1}^{N-1} s_{\alpha_a\,/\,\gamma_a} \left(Q_{F,a+1}\, w^{(a+1)}\right) s_{\beta_a\,/\,\gamma_a} \left(Q_{M,a}\, x^{(a)}\right) s_{\beta_a{}^{t}\,/\,\delta_{a+1}} \\
&\qquad\qquad \left(Q_{M,a+1}\, y^{(a+1)}\right) s_{\alpha_{a+1}{}^{t}\,/\,\delta_{a+1}} \left(Q_{F,a+1}\, z^{(a+1)}\right)
\end{aligned}
$$

Using this result repeatingly, we obtain the second line of the formula (B.1).

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